Ricci-Parallelizable Spaces in the NS-NS Sector:

$AdS_3 \times S^7$

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Abstract

We provide a class of nondilatonic solutions to the NS-NS sector of string theory. The solutions consist of products of Ricci-parallelizable spaces with adjusted radii. A representative of this class, $AdS_3 \times S^7$, is presented in detail. Some comments on possible brane connections are made.

1   Introduction

Central to the current nonperturbative understanding of M-theory has been the prominent role of certain supergravity solutions. Brane solutions were crucial in uncovering dualities between different string theories. The D-brane counterpart of these supergravity solutions helped achieve progress in certain supersymmetric gauge theories. Another class of supergravity solutions that has played an important role has the form $AdS_d \times S^{D-d}$, where $D = 10, 11$. These solutions are central in the AdS/CFT correspondence. A particularly attractive feature of a subclass of these solutions is the fact that the dilaton is constant, since this allows for far more control in the CFT side.

Two solutions that have been missing in the class of nondilatonic $AdS_d \times S^{D-d}$ were the cases $d = 3$ and $d = 7$. Here we provide such solutions as representatives of a class of solutions we outlined. The relevance of these nondilatonic AdS backgrounds to the AdS/CFT remains to be determined but it seems sensible to expect that these solutions will play an important role.

A natural generalization of the AdS/CFT correspondence consists of replacing S^d by appropriate spaces that are less symmetric than spheres, the typical example is replacing S^5 by $T^{1,1}$ [1]. This generalization allows for a richer structure in the CFT side. Some of the solutions belonging to the class we discuss in this paper allow for naturally substituting the sphere for less symmetric manifolds.

The fact that parallelizable spaces have a distinguished position in the context of string theory has been known since the 80's when most of the sigma model analysis took place [2, 3]. It was realized then that parallelizable spaces satisfy the conformal invariance equations. There was, however, a negative feature associated with the fact that in general their central charge is not vanishing. The fact that the central charge is proportional to the scalar curvature imposes a very restrictive condition for critical models, $R = 0$. This condition will naively lead us to physically uninteresting situations.

In this paper we show how to enforce $R = 0$ in a class of physically interesting solutions. Basically, we consider 10D spaces that are products of parallelizable spaces. This ansatz provides a solution to the NS-NS sector of string theory provided we adjust the radii of the two manifolds involved to enforce $R = 0$. A representative of this class, $AdS_3 \times S^7$, is treated in detail in section 3 after outlining the general argument in section 2. Section 4 contains some comments on possible brane interpretations of solutions containing AdS_3 as a factor in the near-horizon geometry. Conclusions are drawn in section 5.

2   Parallelizability in the NS-NS sector

Consider the NS-NS sector of string theory given by the following action[1]

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g}\, e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12}H^2 \right). \tag{1}$$

We are going to specialize to the case of $\phi = const$. The equation of motions are

$$\partial_M(\sqrt{-g}H^{MRS}) = 0,$$

---

[1]A more fundamental view of this analysis can be presented from the sigma-model point of view [2, 3]. Here, however, we are going to concentrate on the spacetime counterpart assuming $D = 10$.

$$R_{MN} = \frac{1}{4} H_{MPQ} H_N^{PQ},$$

$$R = \frac{1}{12} H^2 \tag{2}$$

Here the third line is a constraint coming from the dilaton equation. Noting that the second line implies $R = H^2/4$, one sees that the only solution is $R = H^2 = 0$.

To make the contact with parallelizable spaces more evident, recall that the generalized Riemann tensor is the Riemann tensor calculated from the generalized connection $\hat{\Gamma}^\lambda_{\ \mu\nu} = \Gamma^\lambda_{\ \mu\nu} - \frac{1}{2} H^\lambda_{\ \mu\nu}$,

$$\hat{R}_{\alpha\beta\gamma\rho} = R_{\alpha\beta\gamma\rho} + \frac{1}{2} \nabla_\gamma H_{\alpha\beta\rho} - \frac{1}{2} \nabla_\rho H_{\alpha\beta\gamma} + \frac{1}{4} H_{\sigma\alpha\gamma} H^\sigma_{\ \rho\beta} - \frac{1}{4} H_{\sigma\alpha\rho} H^\sigma_{\ \gamma\beta},$$

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{4} H_{\alpha\gamma\rho} H_\beta^{\ \gamma\rho} + \frac{1}{2} \nabla^\gamma H_{\gamma\alpha\beta} \tag{3}$$

A space is parallelizable if $\hat{R}_{\alpha\beta\gamma\rho} = 0$ and Ricci-parallelizable if $\hat{R}_{\alpha\beta} = 0$. The second equation makes the relevance to the equations of motion of the NS-NS sector of string theory with a constant dilaton explicit. We now have that the Einstein equation is nothing but imposing the vanishing of the symmetric part of the generalized Ricci tensor and that the equation of motion for the NS-NS two-form field is nothing but the vanishing of the antisymmetric part. This means that any Ricci-parallelizable space furnishes us with a potential solution. One needs, however, to take into consideration the constraint coming from the dilaton equation of motion $R = 0$. For this purpose we consider a 10D space that is the direct product of two spaces, in other words, consider splitting the index M into (μ, m) with $\mu = 0, \ldots, d$ and $m = d+1, \ldots, 9-d$ with $x^M = (x^\mu, y^m)$ and the following ansatz for the metric and antisymmetric tensors

$$g_{\mu n} = 0, \quad g_{\mu\nu} = g_{\mu\nu}(x), \quad g_{mn} = g_{mn}(y),$$

$$B_{\mu n} = 0, \quad B_{\mu\nu} = B_{\mu\nu}(x), \quad B_{mn} = B_{mn}(y). \tag{4}$$

Under these assumptions, all we have to do is to adjust by hand the "radii" of the subspaces to guarantee that the total scalar curvature is zero. The situation is strikingly similar to the Freund-Rubin compactification [4]. Here we have that any pair of *Ricci-parallelizable* spaces with opposite scalar curvature provides us with a solution. Note that in the FR compactification all one needs is a pair of Einstein spaces with opposite scalar curvature. Actually, the solutions presented here are more similar to Englert's generalization [5] of the FR compactification in the sense that the antisymmetric tensor is nonvanishing in both subspaces. It is precisely in this sense that the construction presented here generalizes part of that of [6]. The $AdS_7 \times S^3$ background that can be constructed based on the scheme described above has nontrivial three-form tensor in both factors of the space.

3 Examples

The first, now standard, result on parallelizable manifolds was obtained by Cartan and Schouten [7] (see also [8] for a modern discussion). It states that only group manifolds and

S^7 admit an *absolute parallelism*, i.e., are globally parallelizable in a way that leaves the geodesics of the manifold unaltered (with a totally antisymmetric torsion). This classic work was generalized in [9] to pseudo-Riemannian spaces. A particularly interesting generalization to homogeneous spaces was obtained in the series [10] [2]. An earlier discussion of these homogeneous spaces was conducted in [15, 16] from a different point of view. For a partial list of parallelizable spaces the reader is referred to [15, 16][3]. Some common representatives are group manifolds, Stiefel manifolds with the exception of spheres,[4] and G/T where T is a non-maximal toral subgroup.

A proposition by Wolf [9] clarifies why parallelizable spaces are of such interest in various areas of mathematics. It states that for M a connected differentiable manifold, there are natural one-one correspondences between (i) absolute parallelisms ϕ on M; [5] (ii) smooth trivializations of the frame bundle $B \to M$; (iii) smooth connections Γ on $B \to M$ with holonomy group reduced to the identity. This proposition shows that the problem can be tackled using methods of differential geometry, algebraic topology or representation theory.

It is worth noting that any parallelizable space is necessarily Ricci-parallelizable but the converse need not be true. Much of the literature, both in physics and mathematics, has concentrated on parallelizable spaces but we will keep in mind that the condition for being a solution is weaker. Two examples of spaces that are Ricci-parallelizable but are not parallelizable are the squashed S^7 and a particular embedding of $(SU(2) \times SU(2))/U(1)$.

It is also possible to consider more than two factors. The scheme outlined above works perfectly well for the product of any number of Ricci-parallelizable spaces. Recalling that the only parallelizable spheres are S^1, S^3 and S^7 one could form: $AdS_3 \times S^3 \times S^3 \times S^1$ or for that matter $AdS_3 \times S^3 \times T^4$, where T^4 is trivially parallelizable and $AdS_3 \times S^3 \times K3$, where $K3$ is Ricci flat and therefore Ricci parallelizable with trivial torsion. The latter model is related to the D1/D5 system and has received much attention recently.

3.1 $AdS_3 \times S^7$

To make the above description more precise let us consider the following background for the AdS_3 part

$$
\begin{aligned}
ds^2 &= R_1^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + dx^2) \right), \\
B_{tx} &= R_1^2 u^2.
\end{aligned}
\tag{5}
$$

Some properties of this background are

$$
R_{ab} = -\frac{2}{R_1^2} g_{ab}, \quad R = -\frac{6}{R_1^2}, \quad H_{utx} = 2R_1^2 u.
\tag{6}
$$

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[2]D. Lüst explicitly constructed the parallelizing torsion for a class of homogeneous coset spaces [11]. Other explicit examples have been considered recently in [12, 13]

[3]Most of the classification has been carried out under restrictive conditions, such as, for example, assuming that G is simple. The general classification for any G is a more difficult question which depends more intricately on concrete embeddings of H into G [11, 12, 13, 14].

[4]S^1, S^3 and S^7 are the only parallelizable spheres. This fact is related to the corresponding division algebras and to the Hopf fiberings.

[5]This is a statement about vector fields, usually phrased as spaces that can be nonsingularly combed.

Note that the parallelizing torsion is proportional to the product of the natural dreibein $R_1^{-1}(R_1/u)(R_1 u)^2$. This fact is not very revealing at this point since this is basically the only three-form in AdS_3, but in S^7 the situation will be much different. One can check, using these explicit relations, that it satisfies the equations of motion.

The seven sphere and its parallelizing torsion has received a great deal of attention in the 11D supergravity context. Englert [5] used the dual of the parallelizing torsion to generalized the FR solution. The relation of this parallelizing torsion to the octonions was made explicit, for example, in [17]. Here we are going to restrict ourselves to the simplest case of the round S^7. For a more detailed analysis, including the squashed S^7, the reader should consult [18]; for the physical relevance of torsion in the supergravity context see [19]. The following analysis follows most closely references [18, 20, 21]. We consider the following metric on S^7

$$
\begin{aligned}
ds^2 &= R_2^2 \left(d\mu^2 + \frac{1}{4}\sin^2\mu(\sigma_i - \Sigma_i)^2 + (\cos^2\frac{\mu}{2}\sigma_i + \sin^2\frac{\mu}{2}\Sigma_i)^2 \right), \\
\sigma_1 &= \cos\psi_1 d\theta_1 + \sin\psi_1\sin\theta_1 d\phi_1, \\
\sigma_2 &= -\sin\psi_1 d\theta_1 + \cos\psi_1\sin\theta_1 d\phi_1, \\
\sigma_3 &= d\psi_1 + \cos\theta_1 d\phi_1,
\end{aligned}
\tag{7}
$$

similar relations define Σ_i but with the subindex 1 replaced by 2. Following [18, 20, 21], one naturally introduces the following Siebenbein

$$
e^0 = R_2 d\mu, \quad e^i = \frac{R_2}{2}\sin\mu(\sigma_i - \Sigma_i), \quad e^{\hat{i}} = R_2(\cos^2\frac{\mu}{2}\sigma_i + \sin^2\frac{\mu}{2}\Sigma_i),
\tag{8}
$$

where $i, \hat{i} = 1, 2, 3$. In this orthonormal frame the parallelizing torsion is simply proportional to the octonionic multiplication table. It will be more suggestive to call the values of the parallelizing three-form in the orthonormal frame the octonionic structure constant. This is simply the result of Cartan-Schouten [7] that is discussed in [8]: the parallelizing torsion in the orthonormal frame is given by the structure constant.

For imaginary octonions one has

$$
O_a O_b = -\delta_{ab} + f_{abc} O_c, \quad f_{0i\hat{j}} = -\delta_{ij}, \quad f_{ij\hat{k}} = -\epsilon_{ijk}, \quad f_{\hat{i}\hat{j}\hat{k}} = \epsilon_{ijk}
\tag{9}
$$

More precisely one has $H_{abc} = R_2^{-1} f_{abc}$. In the orthonormal frame we obtain

$$
R_{ab} = \frac{3}{2R_2^2}\delta_{ab}, \quad H_{ab}^2 = \frac{6}{R_2^2}\delta_{ab},
\tag{10}
$$

from which the equations of motions follow automatically. Our last task is to find the relation between the radii of AdS_3 and S^7 that makes the total scalar curvature vanish,

$$
R = R_{AdS_3} + R_{S^7} = \frac{-6}{R_1^2} + \frac{21}{2R_2^2} = 0 \Rightarrow \frac{R_1}{R_2} = \frac{2}{\sqrt{7}}.
\tag{11}
$$

It is worth stressing a rather unique property of this solution. The background we are considering is an exact string solution in the sense that it does not receive α' corrections. The AdS_3 part is simply a WZW model on $SL(2, R)$ and although S^7 is not a group manifold, there is a CFT structure defined on it [22] that parallels the WZW construction. This

construction is very similar to the WZW model in the sense that the energy-momentum tensor is given by the Sugawara construction using the currents that generate the associated Kac-Moody algebra. The central charge of this CFT is $c = 7k/(k+12)$ where k is the level of the KM algebra. In the semiclassical approximation that we discussed here, $k = 1/\alpha' \to \infty$ and $c = 7$.

4 Relation to Branes

One question that naturally arises is what is the relation of this class of solutions to strings (1-branes). Although a general analysis is possible we will concentrate on backgrounds possibly having AdS_3 as a factor in the near-horizon geometry. For that purpose we consider the following ansatz

$$
\begin{aligned}
ds^2 &= e^{2A(r)}(-dt^2 + dx^2) + e^{2B(r)}dr^2 + e^{2C(r)}g_{mn}dy^m dy^n, \\
B_{tx} &= e^{C_1(r)}, \quad B_{mn} = e^{C_2(r)}b_{mn}, \quad \phi = \text{const.},
\end{aligned}
\tag{12}
$$

where b_{mn} generates the parallelizing torsion h_{mnp} on a seven-dimensional manifold with metric g_{mn}. At this point, even without further calculation, we draw certain conclusions about the strings and their near-horizon geometry. Note that if we want a "round" AdS_3 in the near horizon limit, $A(r)$ must be related to $B(r)$ in an obvious way to give the same radius for the whole AdS_3. On the other hand, for the typical brane ansatz for the transverse part one has $C(r) = B(r) + \ln r$. This means that the near-horizon geometry of the standard string can be reached only for a 7D having the same radius as AdS_3. More explicitly, in the near-horizon limit to have a "round" AdS_3 we need $A(r) = \ln(rR_1)$ and $B(r) = \ln(R_1/r)$; for the standard string $B(r) = C(r) - \ln r = \ln(R_2/r)$. This means that we can only achieve both conditions (round AdS_3 and standard string ansatz) when the radii of the two parallelizable manifolds are the same ($R_1 = R_2$). This is possible for $AdS_3 \times S^3 \times \mathcal{N}$ with $\mathcal{N}$ being T^4 or $K3$ which is dictated by the "effective" dimensionality of the parallelizable spaces. This simple analysis also shows that $AdS_3 \times S^7$ can not be reached from a standard string ($C(r) = B(r) + \ln r$). The ingredients for the equations of motion are

$$
\begin{aligned}
0 &= \partial_r \left(e^{-2A-B+7C}\mathcal{C}_1 \right), \\
0 &= \partial_r \left(e^{2A-B+3C}\mathcal{C}_2 \right), \\
R_{\mu\nu} &= -\eta_{\mu\nu}e^{2(A-B)}(A'' + 2(A')^2 - A'B'), \\
\frac{1}{4}H^2_{\mu\nu} &= -\frac{1}{2}\eta_{\mu\nu}\mathcal{C}_1^2 e^{-2A-2B}, \\
R_{rr} &= -2(A'' + (A')^2 - A'B') - 7(C'' + (C')^2 - B'C'), \\
\frac{1}{4}H^2_{rr} &= -\frac{1}{2}\mathcal{C}_1^2 e^{-4A} + \frac{1}{2}\mathcal{C}_2^2 e^{-4C}b^2, \\
R_{mn} &= R_{mn}(g) - g_{mn}e^{2(C-B)}(C'' + 7(C')^2 - B'C'), \\
\frac{1}{4}H^2_{mn} &= \frac{1}{4}h^2_{mn}e^{2C_2 - 4C} + \frac{1}{2}\mathcal{C}_2^2 b^2_{mn}e^{-2B-2C},
\end{aligned}
\tag{13}
$$

where $\mathcal{C}_i = (e^{C_i})'$, $b^2 = b_{mn}b_{pq}g^{mp}g^{nq}$, $b_{mn}^2 = b_{mp}b_{nq}g^{pq}$. If we want to engineer a solution for which the Einstein equation for (n, m) is of the form $R_{mn} = h_{mn}^2/4$, we must take $C_2 = 2C$. In order to get rid of the term containing b_{mn}^2 we need $\mathcal{C}_2 = 0$. Altogether this implies that C_2 and C are constants; the term containing g_{mn} also vanishes for constant C.

For constant C_2 and C, the whole system can be uniquely solved with $A' = \frac{q}{2}e^B$. This uniquely fixes the metric to be AdS_3. Namely, going to a new coordinate $dR = e^B dr$ the 3D part of the metric can be written as

$$ds^3 = e^{qR}(-dt^2 + dx^2) + dR^2, \tag{14}$$

which is nothing but AdS_3 in Poincare coordinates. We conclude that, if one insists in a parallelizable manifold as the near-horizon limit of a string, the only solution is AdS_3. It might be natural to expect that $AdS_3 \times S^7$ arises as the limit of branes intersections. A very exhaustive study of brane configurations and the corresponding AdS factor was carried out in [23]. This analysis reveals that the $AdS_3 \times S^7$ geometry does not arise from brane intersections either [23].

It is worth noting that actually for most of the explicit constructions of parallelizable manifolds, the parallelizing torsion in the orthonormal frame is constant. This means that the two-form that generates this torsion should be proportional to the product of vielbeins. This might conspire to produce b_{mn}^2 proportional to g_{mn} and therefore allow for non constant C. Here we will not explore such possibility as it is clear that it depends on the explicit form of b_{mn} but we think it is a very feasible scenario for constructing branes. Another very viable possibility [6] is to generalize the ansatz we considered here to allow $\phi = \phi(r)$ with the condition $\phi = const$ enforced only in the near-horizon limit.

5 Conclusions

We have described a class of solutions to the NS-NS sector of string theory with constant dilaton. Some of the representatives of this class were known from supergravity analysis [13], but some are new. This class of solutions puts a number of particular cases under one unifying scheme, in particular we now recognize the relation between $AdS_3 \times S^3 \times S^3 \times S^1$, $AdS_3 \times S^3 \times T^4$, $AdS_3 \times S^3 \times K3$ and $AdS_3 \times S^7$.

We have described in detail the case of $AdS_3 \times S^7$. We have also discussed the possibility of constructing strings whose near-horizon geometry is of the form $AdS_3 \times \mathcal{N}$ where $\mathcal{N}$ is a Ricci-parallelizable manifold. We found certain restrictions on $\mathcal{N}$ for a specific brane ansatz. In particular we showed that there is no brane solution if one insists that the Einstein equation of motion of $\mathcal{N}$ is strictly the condition of Ricci-parallelizability. We pointed out, however, that it might be possible, by a more detailed analysis, to construct string solutions with the desired near-horizon geometry.

One of the interesting characteristics of this class of solutions is associated with the fact that the NS-NS sector is a universal sector of various string theories. Using U-dualities one could generate backgrounds with R-R fields. This universality property has been exploited in [24], through the use of a web of dualities, to generate new string solutions whose brane worldvolume is a curved space.

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[6]This is the generalization needed for the D1/D5 system

We would like to point out that a general scheme for studying the supersymmetric properties of these solutions is lacking and one would have to deal with it on a case by case basis. In other words, Ricci-parallelizability does not guarantee neither does it forbid preservation of some fraction of the supersymmetry. For example, in the case of $AdS_3 \times \mathcal{N}$ it has been established [13] that in the framework of type IIB some solutions are supersymmetric but some are not. Another feature pointing to the need for a more particular analysis is that the supersymmetry tranformation of these solutions are model-depending. As we have pointed out, these solutions can be embedded in N=2 or N=1 supersymmetric string theories as well as 11D supergravity. This and other related matters will be further discussed in a future publication.

Acknowledgments

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